

A remark on Petersen coloring conjecture of Jaeger

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If G and H are two cubic graphs, then we write $H \prec G$, if G admits a proper edge-coloring f with edges of H , such that for each vertex x of G , there is a vertex y of H with $f(\partial_G(x)) = \partial_H(y)$. Let P and S be the Petersen graph and the Sylvester graph, respectively. In this paper, we introduce the Sylvester coloring conjecture. Moreover, we show that if G is a connected bridgeless cubic graph with $G \prec P$, then $G = P$. Finally, if G is a connected cubic graph with $G \prec S$, then $G = S$.

1. Introduction

The graphs considered here are finite and undirected. They do not contain loops though they may contain multiple edges. For a vertex v of G let $\partial_G(v)$ be the set of edges of G incident to v .

Let G and H be two cubic graphs. Then an H -coloring of G is a proper edge-coloring f with edges of H , such that for each vertex x of G , there is a vertex y of H with $f(\partial_G(x)) = \partial_H(y)$. If G admits an H -coloring, then we will write $H \prec G$.

If $H \prec G$ and f is an H -coloring of G , then for any adjacent edges e, e' of G , the edges $f(e), f(e')$ of H are adjacent. Moreover, if the graph H contains no triangle, then the converse is also true, that is, if a mapping $f : E(G) \rightarrow E(H)$ has a property that for any two adjacent edges e and e' of G , the edges $f(e)$ and $f(e')$ of H are adjacent, then f is a H -coloring of G .

Let P be the well-known Petersen graph (figure 1) and let S be the Sylvester graph (figure 2). Both of them have ten vertices. The Petersen coloring conjecture of Jaeger states:

Conjecture 1 (*Jaeger, 1988 [3]*) *For each bridgeless cubic graph G , one has $P \prec G$.*

The conjecture is difficult to prove, since it can be easily seen that it implies the following two classical conjectures:

Conjecture 2 (*Berge-Fulkerson, 1972 [2]*) *Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings F_1, \dots, F_6 such that any edge belongs to exactly two of them.*

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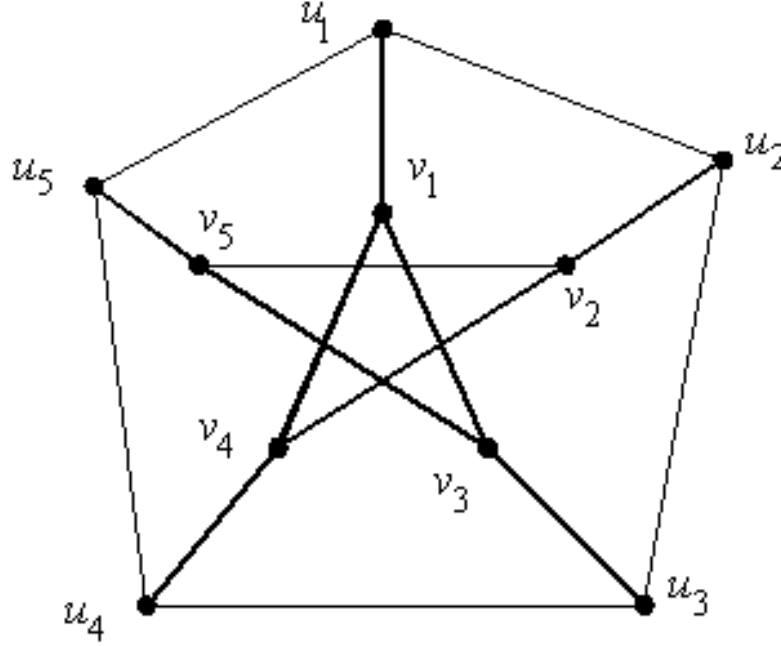


Figure 1. The Petersen graph

Conjecture 3 (*(5,2)-cycle-cover conjecture, [4, 5]*) Any bridgeless graph G (not necessarily cubic) contains five even subgraphs such that any edge belongs to exactly two of them.

Recall that a subgraph H of a graph G is even, if any vertex x of G has even degree in H .

Related with Jaeger conjecture, we would like to introduce the following

Conjecture 4 For each cubic graph G , one has $S \prec G$.

In the direct analogy with Jaeger conjecture, we call this Sylvester coloring conjecture.

One may wonder whether there are other ($\neq P$) bridgeless cubic graphs H , such that for any bridgeless cubic graph G one has $H \prec G$? Similarly, we can look for other ($\neq S$) cubic graphs H , such that for any cubic graph G one has $H \prec G$. It is easy to see that there are infinitely many disconnected bridgeless cubic graphs H meeting this condition provided that Jaeger conjecture is true (hint: take any disconnected bridgeless cubic graph H , which contains a connected component that is isomorphic to the Petersen graph). A similar construction works with Sylvester graph. Thus it is natural to re-state these questions as follows:

Problem 1 Are there other ($\neq P$) connected bridgeless cubic graphs H , such that for any bridgeless cubic graph G one has $H \prec G$ provided that Jaeger conjecture is true?

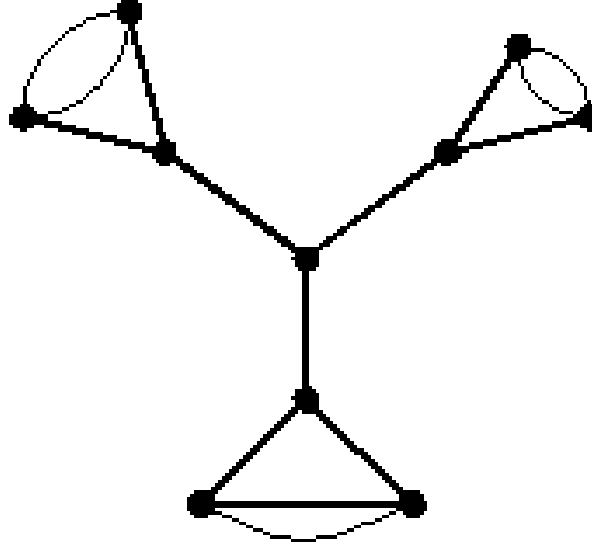


Figure 2. The Sylvester graph

Problem 2 Are there other ($\neq S$) connected cubic graphs H , such that for any cubic graph G one has $H \prec G$ provided that Sylvester coloring conjecture is true?

It is easy to see that the theorems 1 and 2 proved below imply that the answers to these problems are negative.

Non-defined terms and concepts can be found in [1].

2. The main results

For the proof of the main results, we will need the following:

Proposition 1 Let G be a non 3-edge-colorable bridgeless cubic graph that has at most ten vertices. Then $G = P$.

Proposition 2 Let G be a cubic graph that has no a perfect matching and has at most ten vertices. Then $G = S$.

Lemma 1 Suppose that G and H are cubic graphs with $H \prec G$, and let f be an H -coloring of G . Then:

(A) If M is any matching of H , then $f^{-1}(M)$ is a matching of G ;

(B) $\chi'(G) \leq \chi'(H)$, where $\chi'(G)$ is the chromatic index of G ;

(C) If M is any perfect matching of H , then $f^{-1}(M)$ is a perfect matching of G .

Proof. (A) is trivial.

(B) Let $\chi'(H) = l$. Then

$$E(H) = M_1 \cup \dots \cup M_l,$$

where for $i = 1, \dots, l$ M_i is a matching. This implies that:

$$E(G) = f^{-1}(M_1) \cup \dots \cup f^{-1}(M_l).$$

Now, by (A), we have that for $i = 1, \dots, l$ $f^{-1}(M_i)$ is a matching. Thus, G is l -edge-colorable.

(C) By (A), we have that $f^{-1}(M)$ is a matching. Let v be any vertex of G . Since M is a perfect matching of H , we have $f(\partial_G(v)) \cap M \neq \emptyset$. Thus $f^{-1}(M)$ is a perfect matching. \square

We are ready to prove:

Theorem 1 *If G is a connected bridgeless cubic graph with $G \prec P$, then $G = P$.*

Proof. By (B) of lemma 1 G is non 3-edge-colorable. Let f be a G -coloring of P . If $e \in E(G)$, then we will say that e is used (with respect to f), if $f^{-1}(e) \neq \emptyset$.

First of all, let us show that if an edge e of G is used, then any edge adjacent to e , is also used.

So let $e = uv$ be a used edge of G . For the sake of contradiction, assume that v is incident to an edge $z \in E(G)$ that is not used. Suppose that $\partial_G(u) = \{a, b, e\}$. Observe that a and b are also used.

Due to symmetry of Petersen graph, we can assume that $f(u_3u_4) = e$. Suppose that $f(u_4u_5) = a$ and $f(u_4v_4) = b$ (figure 1). Since the edge z is not used, we have: $f(\partial_P(u_3)) = \partial_G(u) = \{a, b, e\}$. Again, due to symmetry of Petersen graph, we can assume that $f(u_3v_3) = b$ and $f(u_2u_3) = a$.

Let $a_1 = f(u_1u_5)$, $a_2 = f(u_1u_2)$. Observe that since f is a G -coloring of P , we have that a_1 and a_2 are adjacent edges of G . Moreover, each of them is adjacent to a . Similarly, the edges $b_1 = f(v_1v_4)$ and $b_2 = f(v_1v_3)$ of G are adjacent, and each of them is adjacent to b .

We will differ three cases:

Case 1: The edges a_1 , a_2 and a do not form a triangle in G .

Observe that in this case $f(u_1v_1) = a$. This implies that the edges a , b_1 , b_2 must be incident to the same vertex. However, this is possible only when b_1 and b_2 are two parallel edges connecting the other ($\neq u$) end-vertices of edges a and b , which is a contradiction, since e cannot be a bridge.

Case 2: The edges b_1 , b_2 and b do not form a triangle in G .

This case is similar to case 1.

Case 3: The edges a_1 , a_2 and a form a triangle in G . Similarly, b_1 , b_2 and b form a triangle.

Let a_3 be the edge of G that is adjacent to a_1 , a_2 and is not adjacent to a . Note that such an edge exists since G is bridgeless. Similarly, let b_3 be the edge of G that is adjacent to b_1 , b_2 and is not adjacent to b . Observe that $a_3 = f(u_1, v_1) = b_3$, and hence $a_3 = b_3$. On the other hand, since G is bridgeless and $a \neq b$, we have $a_3 \neq b_3$, which is a contradiction.

We are ready to complete the proof of theorem 1. Observe that since G is connected, we have that all edges of G are used, and hence $|E(G)| \leq |E(P)|$, or $|V(G)| \leq |V(P)| = 10$. Proposition 1 implies that $G = P$.

□

Theorem 2 *Let G be any connected cubic graph with $G \prec S$. Then $G = S$.*

Proof. Let G be a connected cubic graph with $G \prec S$, and let f be the corresponding coloring. Clearly, G has no a perfect matching (see (C) of lemma 1).

Again, an edge $e \in E(G)$ is used (with respect to f) if $f^{-1}(e) \neq \emptyset$.

First of all let us show that if an edge of G is used, then all edges adjacent to it are used, too. Suppose that $a = uv$ is a used edge of G that is adjacent to a non-used edge. Suppose that v is incident to a non-used edge. Let b and c be the other edges incident to u .

Since a is used, there is $e \in E(S)$ with $f(e) = a$. We will consider three cases.

Case 1: The multiplicity of e is two in S . Suppose that the edge parallel to e is colored by b . Then the two edges of S forming a triangle with e must be colored with c which is impossible.

Case 2: e is adjacent to an edge of multiplicity two in S . Then the two parallel edges must be colored with b and c , and hence the other edge that is adjacent to the same two parallel edges must be colored with a , which is again a contradiction.

Case 3: e is a bridge in S . Let e' and e'' be two non-bridge edges of S that are adjacent to e . We can assume that $f(e') = b$ and $f(e'') = c$. Finally, let g and h be the two parallel edges adjacent to e' and e'' . By Case 1, g and h cannot be colored by a , hence the colors of g and h must be adjacent to both b and c . It is clear that if x and y denote the colors of g and h , then x and y must form a multi-edge in G .

Let z be the bridge of S colored by b . By Case 2, none of the edges adjacent to z and non-adjacent to e , can be colored by a , hence these two edges are colored x and y . Now, observe that the two parallel edges adjacent to these two edges must be colored by b and y , or c and y , which is a contradiction.

To complete the proof, let us note that since G is connected, the proved property implies that all edges of G are used, hence $|E(G)| \leq |E(S)| = 15$ or $|V(G)| \leq |V(S)| = 10$. Proposition 2 implies that $G = S$. \square

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